MATH266: Final Exam, May 12, 2023

Instructions

- Start each problem on a new page;
- Show all your work for partial credit;
- Box your answer;
- No calculators allowed;
- A standard 8 1/2 by 11 sheet of paper with student's notes (both sides) is allowed.

Problems

1. Solve the following IVP

$$(a) \quad y' = -2y,\ y(0) = 3; \quad (b) \quad y' + y = t,\ y(0) = 4.$$

2. Consider a mass of $4kg$ on the spring with the spring constant $k = 100N/m$. Assuming that there is no damping in the system, (a) determine the equation of motion if the mass is released from $1m$ below the equilibrium position with a downward initial velocity of $10m/s$. (b) What are the period and amplitude of the oscillations? (c) Sketch the phase portrait for the system. Is the origin Lyapunov stable, asymptotically stable, or unstable?

3. Find the general solution to

$$y''' - 10y'' + 25y' = 0.$$

4. Solve using the Laplace transform

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5.$$

5. Find the solution to the IVP

$$\dot{\boldsymbol{y}} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

6. Find the real solution to the IVP

$$\dot{\boldsymbol{y}} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

7. Sketch the phase portraits for the following systems. For each case clearly specify whether the origin is Lyapunov stable, asymptotically stable, or unstable.

$$(a) \quad \text{System from Problem 5}; \quad (b) \quad \dot{\boldsymbol{y}} = \begin{bmatrix} -4 & -1 \\ 1 & -6 \end{bmatrix} \boldsymbol{y}; \quad (c) \quad \text{System from Problem 6}.$$

8. (This problem is optional. If you have time, you can solve it for extra credit.)
   Consider Lanchester's model of combat between the Blue B and the Red R:

$$\dot{R} = -bB, \quad \dot{B} = -rR,$$

   where b, r are the combat efficiency coefficients for the blue and red armies respectively. Assuming that $b = 2$, $r = 10$ and the initial numbers of combatants are $B(0) = 300$ and $R(0) = 100$ determine the side that will eventually win. You need to provide clear mathematical arguments for your conclusion, simple answers like "the Blue will win" or "the Red will win" will not get you any credit.